Scaling laws for the largest Lyapunov exponent in long-range systems: A random matrix approach

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We investigate the laws that rule the behavior of the largest Lyapunov exponent (LLE) in many particle systems with long-range interactions. We consider as a representative system the so-called Hamiltonian α-XY model where the adjustable parameter α controls the range of the interactions of N ferromagnetic spins in a lattice of dimension d. In previous work the dependence of the LLE with the system size N, for sufficiently high energies, was established through numerical simulations. In the thermodynamic limit, the LLE becomes constant for $\alpha > d$ whereas it decays as an inverse power law of N for $\alpha < d$. A recent theoretical calculation based on a geometrization of the dynamics is consistent with these numerical results. Here we show that the scaling behavior can also be explained by a random matrix approach, in which the tangent mappings that define the Lyapunov exponents are modeled by random simplectic matrices drawn from a suitable ensemble.

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I. INTRODUCTION

Dynamical systems of many particles interacting via long-range forces can exhibit interesting anomalies such as superdiffusion [1–3], metastable states [4], and non-Boltzmann-Gibbs distribution functions [5]. Special interest in such systems has arisen recently in connection with the nonextensive generalization of statistical mechanics introduced by Tsallis [6].

The existence of a dynamics makes the systems mentioned above very attractive because it is possible, in principle, to associate the properties of the thermodynamic states with features of the many particle phase space. As a remarkable example, let us mention the ''topological hypothesis,'' which relates thermodynamic phase transitions to topological changes in the structure of phase space [7].

A dynamical model with an adjustable interaction range α, and allowing extensive numerical and analytical exploration, has been recently introduced [8–10]. The model consists of a periodical d-dimensional lattice of N interacting rotators moving on parallel planes. Each rotator is restricted to the unit circle and therefore it is fully described by an angle $0 < \theta_i \leq 2\pi$ and its conjugate momentum L_i, with $i = 1, \ldots, N$. The dynamics of the system is governed by the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} L_i^2 + \frac{J}{2}\sum_{i,j=1, i\neq j}^{N} \frac{1-\cos(\theta_i-\theta_j)}{r_{ij}^{\alpha}}, \qquad (1)$$

where the coupling constant is $J \geq 0$ and, without loss of generality, unitary moments of inertia are chosen for all the particles. Here r_{ij} measures the minimal distance between the rotators located at the lattice sites i and j. The Hamiltonian (1) describes a classical inertial XY ferromagnet. It contains as particular cases the mean-field version ($\alpha/d = 0$) and the first-neighbors case, recovered in the $\alpha/d \to \infty$ limit.

The systems α-XY characterized by interaction ranges $0 \leq \alpha/d < 1$ do not have a well defined thermodynamic limit, e.g., the specific (per particle) energy diverges when $N \to \infty$. A proper thermodynamic limit is defined by introducing a scaling parameter $\tilde{N}$ [11], which depends on N, α, and d [12,13]:

$$\tilde{N} = \frac{1}{N}\sum_{i,j=1, i\neq j}^{N} \frac{1}{r_{ij}^{\alpha}}. \qquad (2)$$

In the large N limit one has

$$\tilde{N}(\alpha/d) \sim \begin{cases} N^{1-\alpha/d}, & 0 \leq \alpha < d \\ \ln N, & \alpha = d \\ \Theta(\alpha/d), & \alpha > d \end{cases} \qquad (3)$$

with Θ a function of the ratio α/d only. Specific energylike quantities must be rescaled by $\tilde{N}$. At the dynamic level, time, hence inverse Lyapunov exponents, have to be scaled by $\tilde{N}^{-1/2}$ [8].

A completely equivalent description is obtained by working with the already scaled Hamiltonian

$$\tilde{H} = \frac{1}{2}\sum_{i=1}^{N} L_i^2 + \frac{J}{2\tilde{N}}\sum_{i,j=1, i\neq j}^{N} \frac{1-\cos(\theta_i-\theta_j)}{r_{ij}^{\alpha}}. \qquad (4)$$

This kind of scaling of the strength of the interactions, common in standard mean-field discussions, has been applied to the study of the mean-field case of the present model in Refs. [14,15]. Since the Hamiltonian (4) leads to the same results as Eq. (1), but avoiding further rescalings, all our considerations from here on will be related to the already scaled Hamiltonian (4).

The α-XY ferromagnet has been subject of several numerical and analytical studies. References [8,12,13] and [9,10] are dedicated to the cases $d = 1$ and $d = 2,3$, respectively. The mean-field problem is discussed in Refs. [14,15] while the opposite limit of first-neighbor interactions and $d = 1$ can be found in Ref. [16]. The one-dimensional case has

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been explored more extensively: For long-ranged interactions, $\alpha<1$, the system displays a second order phase transition from a ferromagnetic state to a paramagnetic one at a certain critical specific energy ε_c ($\varepsilon_c=0.75J$) [12]. Systems with $1<\alpha<2$ also undergo a second order phase transition but the critical energy depends on α [12]. For $\alpha>2$, all systems behave similarly to the first-neighbor model, where there is no order nor phase transition for finite energies in the thermodynamic limit [12].

Here we are concerned with the high energy phase, i.e., with energies above any critical energy that the system may have. In this disordered regime, the kinetic energy is much larger than the bounded potential energy, the rotators are weakly coupled, and the dynamics is weakly chaotic. In Refs. [8–10] the largest Lyapunov exponents (LLE's) were calculated numerically. It was found that, in the thermodynamic limit and for large enough energies, the largest Lyapunov exponent remains positive and finite for short-range interactions ($\alpha/d>1$) but vanishes as an inverse power law of the system size in the long-range case $0 \leqslant \alpha/d<1$.

Recently Pettini and co-workers developed a theoretical method which, in principle, allows us to obtain the scaling behavior of the LLE analytically [17]. In this approach the phase space trajectories are mapped onto a geodesic flow in configuration space (equipped with a suitable metric). It is assumed that the curvature fluctuations along an ergodic geodesic can be modeled as a Gaussian process. Then the LLE is expressed in terms of the mean and variance of the process. These parameters are calculated as microcanonical averages of suitable dynamical functions. There are several works where the method was applied to the α-XY model. The scaling behavior of the LLE in the extreme cases $\alpha\to\infty$ and $\alpha=0$ was found in Refs. [18] and [19], respectively. Very recently, Firpo and Ruffo [20] succeeded in calculating the LLE scaling laws for any interaction range $0\leqslant\alpha/d\leqslant1$.

It is our purpose here to present a simple alternative procedure, based on a random matrix formulation, which allows one to derive the dependence of the LLE on the size of the system N, the range of the interactions α, and the lattice dimension d. This procedure is based on the ideas introduced by Benettin [21] in the discussion of two-dimensional billiards, and later extended to interacting many particle systems [22,23].

II. THE LARGEST LYAPUNOV EXPONENT

The equations of motion generated by the Hamiltonian (4) are

$$\dot{\theta}_i = L_i,$$

$$\dot{L}_i = -\frac{J}{\widetilde{N}} \sum_{j=1,j\neq i}^{N} \sin(\theta_i-\theta_j)/r_{ij}^{\alpha} \tag{5}$$

for $i=1,\ldots,N$. Discretizing the time axis into steps Δt one obtains the stroboscopic map relating angles and momenta at successive discrete times

$$\theta_i' = \theta_i + L_i\Delta t,$$

$$L_i' = L_i - \frac{J}{\widetilde{N}} \sum_{j=1,j\neq i}^{N} \sin(\theta_i'-\theta_j')/r_{ij}^{\alpha}\Delta t. \tag{6}$$

For the purpose of discussing Lyapunov exponents one has to consider the tangent map T, i.e., the linearized version of Eqs. (6), given by

$$\delta\theta_i' = \delta\theta_i + \delta L_i\Delta t,$$

$$\delta L_i' = \delta L_i - \frac{J}{\widetilde{N}} \sum_{j=1,j\neq i}^{N} \cos(\theta_i'-\theta_j')[\delta\theta_i'-\delta\theta_j']/r_{ij}^{\alpha}\Delta t. \tag{7}$$

In matrix form these equations read

$$\begin{pmatrix} \delta\boldsymbol{\theta}' \\ \delta\boldsymbol{L}' \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \mathbb{1}\Delta t \\ \boldsymbol{\epsilon}\hat{b}\Delta t & \mathbb{1}+\boldsymbol{\epsilon}\hat{b}(\Delta t)^2 \end{pmatrix} \begin{pmatrix} \delta\boldsymbol{\theta} \\ \delta\boldsymbol{L} \end{pmatrix} \equiv T \begin{pmatrix} \delta\boldsymbol{\theta} \\ \delta\boldsymbol{L} \end{pmatrix}, \tag{8}$$

where all submatrices are of size $N\times N$, $\mathbb{1}$ being the identity. The matrix $\hat{b}$ is given by

$$b_{ij} = \begin{cases} \cos(\theta_i'-\theta_j')/r_{ij}^{\alpha} & \text{for} \quad j\neq i, \\ -\sum_{k=1,k\neq i}^{N} \cos(\theta_i'-\theta_k')/r_{ik}^{\alpha} & \text{for} \quad j=i, \end{cases} \tag{9}$$

and the perturbation parameter $\boldsymbol{\epsilon}$ is

$$\boldsymbol{\epsilon} = J/\widetilde{N}(\alpha/d). \tag{10}$$

For long-range interactions it is clear that $\boldsymbol{\epsilon}$ goes to zero in the thermodynamic limit. In the short-range case one has $\boldsymbol{\epsilon}\sim J$. In order to treat $\boldsymbol{\epsilon}$ as a small parameter in both cases, we will consider the limit $J\to0$ when necessary.

The LLE can be defined by the limiting procedure

$$\lambda_{max} = \lim_{n\to\infty} \frac{1}{n\Delta t} \ln\|\mathcal{T}\xi\|, \tag{11}$$

with ξ an arbitrary vector and $\mathcal{T}\equiv T_nT_{n-1}\cdots T_2T_1$ is the product of n tangent maps calculated at successive points of the discretized trajectory. Using the Euclidean norm, Eq. (11) is rewritten as

$$\lambda_{max} = \lim_{n\to\infty} \frac{1}{2n\Delta t} \ln(\xi^t\mathcal{T}^t\mathcal{T}\xi), \tag{12}$$

the superscript t indicating ''transposed.''

III. RANDOM MATRIX APPROACH

The random matrix approach is based in the modeling of the tangent mappings T_k by a sequence of noncorrelated random simplectic matrices mimicking the essential properties of the chaotic dynamics. In the standard procedure [22] one replaces the short-time T_k by finite time random matrices R_k having the same structure,

$$R_k=\begin{pmatrix} \mathbb{1} & \mathbb{1}\tau \\ \epsilon\hat{a}_k\tau & \mathbb{1}+\epsilon\hat{a}_k\tau^2 \end{pmatrix}. \tag{13}$$

In previous treatments the time scale τ has always been ignored by setting it to 1. Here we prefer to keep track of τ, as later we will argue that it is related to energy. It is fixed as follows. The time scale τ must be chosen small enough in order to preserve the short time structure of the tangent maps. However, it cannot be too small, as we assume that consecutive tangent maps are statistically independent. Thus τ must be an intermediate scale, of the order of the correlation time for the tangent maps. (This time scale is analogous to the correlation time of the Gaussian process modeling the fluctuations of the curvature in the geometric method [17].)

The symmetric matrices $\hat{a}_k$ are the random analogs of $\hat{b}_k$ and are not correlated, i.e., $\langle\hat{a}_n\hat{a}_m\rangle=\langle\hat{a}_n\rangle\langle\hat{a}_m\rangle$, unless $n=m$. Except for the symmetry restriction, the elements a_{ij} of a given matrix $\hat{a}_k$ are assumed to be independent [24].

The probability distributions of the elements a_{ij} are obtained from b_{ij} (9) by considering that the angles θ_i' are independent from each other and uniformly distributed in $[0,2\pi)$. This assumption, reasonable in the high energy phase, implies that: (i) The average of each a_{ij} is null. (ii) The information about the range of the interactions is embodied in the variance of each a_{ij}, which depends on the distance between sites:

$$\langle(a_{ij})^2\rangle=\frac{1}{2}r_{ij}^{-2\alpha}\quad(i\neq j),$$

$$\langle(a_{ii})^2\rangle=\frac{1}{2}\tilde{N}(2\alpha/d). \tag{14}$$

In this way one has defined a crude although nontrivial statistical model whose validity has been shown in previous works [21–23].

As a consequence of the assumptions made one has the property $\langle\hat{a}_k^t\hat{a}_k\rangle=\gamma\mathbb{1}$, which will be useful for evaluating averages. In our case,

$$\gamma=\tilde{N}(2\alpha/d). \tag{15}$$

Within this model the expression for the LLE is obtained by averaging over different realizations of sequences of random matrices

$$\lambda_{max}=\lim_{n\to\infty}\frac{1}{2n\tau}\langle\ln\xi^tR_1^t\cdots R_{n-1}^tR_n^tR_nR_{n-1}\cdots R_1\xi\rangle. \tag{16}$$

Assuming that the distribution of LLE's over the ensemble of sequences is narrow, one can interchange the average and the logarithm. Then the averaging scheme is reduced to a sequence of averages over each matrix distribution,

$$\lambda_{max}\simeq\lim_{n\to\infty}\frac{1}{2n\tau}\ln(\xi^t\langle\cdots\langle R_{n-1}^t\langle R_n^tR_n\rangle R_{n-1}\rangle\cdots\rangle\xi). \tag{17}$$

These averages have already been calculated by Parisi and Vulpiani [23] (see also Ref. [25]). Instead of just recalling their results, we prefer to exhibit a different derivation, which not only makes the paper self-contained, but may be interesting by itself. Notice that the first average can be immediately done, the result being the symmetric matrix

$$\langle R_n^tR_n\rangle=\begin{pmatrix} \nu_1\mathbb{1} & \sigma_1\mathbb{1} \\ \sigma_1\mathbb{1} & \mu_1\mathbb{1} \end{pmatrix}, \tag{18}$$

with

$$\nu_1=1+\gamma\epsilon^2\tau^2,$$

$$\mu_1=1+\tau^2+\gamma\epsilon^2\tau^4,$$

$$\sigma_1=\tau+\gamma\epsilon^2\tau^3. \tag{19}$$

The remaining $n-1$ averages can be done iteratively, obtaining at each step a symmetric matrix with the same structure as Eq. (18). For instance, the second step consists in calculating $\langle R_{n-1}^t\langle R_n^tR_n\rangle R_{n-1}\rangle$. After performing the matrix product, the average is done using that $\langle\hat{a}_{n-1}\rangle=0$ and $\langle\hat{a}_{n-1}^t\hat{a}_{n-1}\rangle=\gamma\mathbb{1}$. The result is

$$\left\langle R_{n-1}^t\begin{pmatrix} \nu_1\mathbb{1} & \sigma_1\mathbb{1} \\ \sigma_1\mathbb{1} & \mu_1\mathbb{1} \end{pmatrix}R_{n-1}\right\rangle=\begin{pmatrix} \nu_2\mathbb{1} & \sigma_2\mathbb{1} \\ \sigma_2\mathbb{1} & \mu_2\mathbb{1} \end{pmatrix}, \tag{20}$$

where

$$\begin{pmatrix} \nu_2 \\ \mu_2 \\ \sigma_2 \end{pmatrix}=\begin{pmatrix} 1 & \gamma\epsilon^2\tau^2 & 0 \\ \tau^2 & 1+\gamma\epsilon^2\tau^4 & 2\tau \\ \tau & \gamma\epsilon^2\tau^3 & 1 \end{pmatrix}\begin{pmatrix} \nu_1 \\ \mu_1 \\ \sigma_1 \end{pmatrix}\equiv M\begin{pmatrix} \nu_1 \\ \mu_1 \\ \sigma_1 \end{pmatrix}. \tag{21}$$

Iterating this procedure up to the last step, we obtain

$$\langle R_1^t\langle\cdots\rangle R_1\rangle=\begin{pmatrix} \nu_n\mathbb{1} & \sigma_n\mathbb{1} \\ \sigma_n\mathbb{1} & \mu_n\mathbb{1} \end{pmatrix}, \tag{22}$$

where the coefficientes are given by

$$\begin{pmatrix} \nu_n \\ \mu_n \\ \sigma_n \end{pmatrix}=M\begin{pmatrix} \nu_{n-1} \\ \mu_{n-1} \\ \sigma_{n-1} \end{pmatrix}=M^{n-1}\begin{pmatrix} \nu_1 \\ \mu_1 \\ \sigma_1 \end{pmatrix}. \tag{23}$$

The LLE is related to the maximum eigenvalue of the matrix (22):

$$\lambda_{max}=\lim_{n\to\infty}\frac{1}{2n\tau}\ln(\nu_n+\mu_n+\sqrt{(\nu_n-\mu_n)^2+4\sigma_n^2}). \tag{24}$$

By virtue of the recurrence relation (23), ν_n, μ_n, and σ_n grow like L_{max}^n, where L_{max} is the maximum eigenvalue of the 3×3 recurrence matrix M. Then

$$\lambda_{max}=\frac{1}{2\tau}\ln L_{max}. \tag{25}$$

After solving the cubic eigenvalue equation we expand L_{max} around $\epsilon=0$,

$$L_{max}=1+2^{1/3}(\gamma\epsilon^2\tau^4)^{1/3}+\frac{1}{2^{1/3}}(\gamma\epsilon^2\tau^4)^{2/3}+\cdots, \tag{26}$$

so that

$$\lambda_{max}=\frac{1}{2^{2/3}}(\gamma\epsilon^2\tau)^{1/3}-\frac{1}{360\times2^{1/3}}(\gamma\epsilon^2\tau)^{5/3}\tau^4+\cdots. \tag{27}$$

Finally, substituting ϵ and γ by their definitions (10) and (15), respectively, one gets the compact expression

$$\lambda_{max}\propto J^{2/3}\tau^{1/3}\left[\frac{\tilde{N}(2\alpha/d)}{\tilde{N}^2(\alpha/d)}\right]^{1/3}. \tag{28}$$

However, for the purpose of comparison with previous works, it is convenient to make explicit the N dependence. Recalling the asymptotic expression (3) for $\tilde{N}$, we arrive at

$$\lambda_{max}\propto J^{2/3}\tau^{1/3}\begin{cases}1/N^{1/3}, & 0\leqslant\alpha/d<1/2\\ (\ln N/N)^{1/3}, & \alpha/d=1/2\\ 1/N^{2(1-\alpha/d)/3}, & 1/2<\alpha/d<1\\ 1/(\ln N)^{2/3}, & \alpha/d=1\\ \text{constant}, & 1<\alpha/d.\end{cases} \tag{29}$$

This scaling law for the LLE can also be written as $\lambda_{max}\sim1/N^{\kappa}$ with

$$\kappa=\begin{cases}1/3, & 0\leqslant\alpha/d\leqslant1/2\\ 2(1-\alpha/d)/3, & 1/2<\alpha/d<1,\\ 0, & 1<\alpha/d\end{cases} \tag{30}$$

the case $\alpha/d=1$ being marginal.

We expect the scaling above not to depend on the details of the dynamics, i.e., it should be typical of systems with couplings of the form $1/r^{\alpha}$, e.g., classical n-vector ferromagnets (of which $n=2$ is the present case), as long as the perturbation $\hat{a}$ has zero mean. Systems for which the average perturbation is nonzero belong to a different universality class, and alternative scaling laws are expected [21–23,25].

IV. CONCLUDING REMARKS

The scaling behavior of λ_{max} with $N\to\infty$ and $J\to0$ [Eq. (29)] is exactly the same as that obtained by using the geometric method [20]. The agreement can be also extended to the energy domain ($\varepsilon\to\infty$) by relating the time scale τ to energy. Given that the potential energy is bounded, when $\varepsilon\to\infty$, the total energy and the kinetic energy are essentially the same. In this regime, changing the time scale is equivalent to a change in the kinetic energy, so that we have $\tau\propto\varepsilon^{-1/2}$. Thus we obtain the scaling law $\lambda_{max}\propto\varepsilon^{-1/6}$.

The theoretical results (30) agree with the numerical ones obtained in Refs. [8–10]. There are some deviations which are consistent with finite size effects, as argued in Ref. [20]. However, one should not discard the possibility that the scaling laws are not exactly those derived in this paper (or in Ref. [20]). The differences with numerical calculations might be due to the fact that both the geometric and the random matrix approaches assume ergodicity and fast (exponential) decay of the correlations. We do not know at present if the dynamical system fully satisfies these hypotheses. Eventually this issue will be decided when simulations on larger systems are available.

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